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Analytical Solution of 1-dimensional Heat Equation by Elzaki Transform

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Abstract

In this work a new Integral transform, namely Elzaki Transform was applied to solve 1-dimensional heat equation to obtained the exact solutions.It is moreeasier, efficient,simple and powerful tool for solving 1-dimensional heat equation. Some examples are solved to show the ability of this method.

Keywords:Partial Differential Equation,Heat Equation, Elzaki Transform.

Introduction

It is common that most of the phenomena in the field of engineering and mathematical physics can be represented by partial differential equations (PDEs). For example, in physics the phenomena of flow of heat and propagation of wave are represented by partial differential equation in a better way. In the field of ecology, most models related to population are described by partial differential equations. The scattering of chemically reacted materials is described by partial differential equation. It is very easy to describe that most of the phenomena governing electricity, quantum mechanics, plasma physics, fluid mechanics, propagation of shallow water wave and many other models lie within the domain of partial differential equations.

There are many integral transforms in the literaturewhich are widely used in engineering, astronomy as well as in physics. The Laplace transform plays versatile (useful) role in its applications among the all transforms.Integral transform is an efficient method to solve the system of ordinary and partial differential equations. Recently, TarigElzaki introduced a new transform, named as Elzaki Transform [5, 6, 7, 8, 9] which is based on Fourier transform and further applied it to the solution of ordinary differential equations [7], telegraph equation [5], system of partial differential equations [8] and integral equations [6].The advantage of this method is that, it solves the problem directly without the need for linearization, perturbation, or any other transformation, and also, reduces the massive computation works required by most other methods [2, 3, 4].

Elzaki Transform

Definition [8]

Elzaki Transform of a function $f(t)$ is denoted by $E[f(t)]$ and is defined as follows

$$E[f(t)] = T(v) = v\int_0^\infty e^{-\frac{t}{v}} f(t)dt \qquad , \qquad v \in (-k_1, k_2)$$

We use integration by parts to obtain Elzaki Transform of partial derivatives as

$$E\left[\frac{\partial f(x,t)}{\partial t}\right] = v\int_0^\infty \frac{\partial f}{\partial t} e^{-\frac{t}{v}} dt = \lim_{m\to\infty} \int_0^m v\frac{\partial f}{\partial t} e^{-\frac{t}{v}} dt$$

$$= \lim_{m\to\infty}\left\{\left[ve^{-\frac{t}{v}}f(x,t)\right]_0^m - \int_0^m e^{-\frac{t}{v}}f(x,t)dt\right\} = \frac{T(x,v)}{v} - vf(x,0)$$

We assume that f is piecewise and of exponential order

$$E\left[\frac{\partial f(x,t)}{\partial x}\right] = v\int_0^\infty \frac{\partial f}{\partial x}e^{-\frac{t}{v}}dt = \frac{\partial}{\partial x}\int_0^m ve^{-\frac{t}{v}}f(x,t)dt$$

Using the Leibnitz rule we find that

$$E\left[\frac{\partial f}{\partial x}\right] = \frac{d}{dx}[T(x,v)]$$

We can also find

$$E\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2}[T(x,v)]$$

The Operation Properties of Elzaki Transform[5, 8, 9]:-

Elzaki Transform of some Functions

Sr. No	$f(t)$	$E[f(t)] = T(v)$
1	1	v^2
2	t	v^3
3	t^n	$n!\ t^{n+2}$
4	e^{at}	$\frac{v^2}{1-av}$
5	te^{at}	$\frac{v^3}{(1-av)^2}$
6	$\frac{t^{n-1}e^{at}}{(n-1)!}$, $n = 1, 2, \ldots$	$\frac{v^{n+1}}{(1-av)^n}$
7	$sinat$	$\frac{av^3}{1+a^2v^2}$
8	$cosat$	$\frac{v^2}{1+a^2v^2}$
9	$sinhat$	$\frac{av^3}{1-a^2v^2}$
10	$coshat$	$\frac{av^2}{1-a^2v^2}$
11	$e^{at}sinbt$	$\frac{bv^3}{(1-av)^2+b^2v^2}$
12	$e^{at}cosbt$	$\frac{(1-av)v^2}{(1-av)^2+b^2v^2}$
13	$tsinat$	$\frac{2av^4}{1+a^2v^2}$
14	$tcosat$	$\frac{v^3}{1+a^2v^2}$

Numerical Applications:

Example 1 [1]:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad 0 < x < 1\,,\ \ 0 \le t \tag{1}$$

With boundary conditions

$$u(0,t) = u(1,t) = 0\ , \quad 0 < t \tag{2}$$

And initial conditions $u(x,0) = \sin(\pi x)$

Solution:

The exact solution is $u(x,t) = e^{-\pi^2 t} \sin(\pi x)$

Taking Elzaki Transform on both sides on (1)& (2)we get

$$\frac{T(x,v)}{v} - vu(x,0) - \frac{d^2 T(x,v)}{dx^2} = 0 \tag{3}$$

$$T(0,v) = T(1,v) = 0 \tag{4}$$

Using initial conditions we get

$$v\frac{d^2 T(x,v)}{dx^2} - T(x,v) = -v^2 \sin(\pi x)$$

$$vT'' - T = -v^2 \sin(\pi x)$$

After solving above differential equation we get

$$T(x,v) = C_1 e^{\frac{x}{\sqrt{v}}} + C_2 e^{-\frac{x}{\sqrt{v}}} + \frac{v^2}{1+\pi^2 v} \sin(\pi x)$$

Using (4) we get

$$T(x,v) = \frac{v^2}{1+\pi^2 v} \sin(\pi x)$$

Taking inverse Elzaki Transform on both sides

$$u(x,t) = \sin(\pi x) E^{-1}\left(\frac{v^2}{1+\pi^2 v}\right)$$

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x)$$

Example 2 [1]:

$$\frac{\partial u(x,t)}{\partial t} - \frac{1}{\pi^2}\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad 0 < x < 1\,,\ \ 0 < t \tag{5}$$

With boundary conditions

$$u(0,t) = u(1,t) = 0\ , \quad 0 < t \tag{6}$$

And initial conditions $u(x,0) = \cos\pi\left(x - \frac{1}{2}\right) \qquad ,0 \le x \le 1$

Solution:

The exact solution is $u(x,t) = e^{-t}\cos\pi\left(x-\frac{1}{2}\right)$

Taking Elzaki Transform on both sides on (5)& (6)and using initial condition we get

$$\frac{v}{\pi^2}\frac{d^2T(x,v)}{dx^2} - T(x,v) = -v^2\cos\pi\left(x-\frac{1}{2}\right) \quad (7)$$

$$T(0,v) = T(1,v) = 0 \quad (8)$$

After solving above differential equation (7) we get

$$T(x,v) = C_1e^{\frac{\pi x}{\sqrt{v}}} + C_2e^{-\frac{\pi x}{\sqrt{v}}} + \frac{v^2}{1+v}\cos\pi\left(x-\frac{1}{2}\right)$$

Using (8) we get

$$T(x,v) = \frac{v^2}{1+v}\cos\pi\left(x-\frac{1}{2}\right)$$

Taking inverse Elzaki Transform on both sides we get

$$u(x,t) = e^{-t}\cos\pi\left(x-\frac{1}{2}\right)$$

Example 3 [4]:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < \pi, \quad 0 < t \quad (9)$$

With boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad 0 < t \quad (10)$$

And initial conditions $u(x,0) = x^2$

Solution:

The exact solution is $u(x,t) = x^2 + 2t$

Taking Elzaki Transform on both sides on (9)& (10)and using initial conditions we get

$$v\frac{d^2T(x,v)}{dx^2} - T(x,v) = -v^2x^2 \quad (11)$$

$$T(0,v) = T(\pi,v) = 0 \quad (12)$$

After solving (11) we get

$$T(x,v) = C_1e^{\frac{x}{\sqrt{v}}} + C_2e^{-\frac{x}{\sqrt{v}}} + v^2x^2 + 2v^3$$

Using (12) we get

$$T(x,v) = v^2x^2 + 2v^3$$

Taking inverse Elzaki Transform on both sides

$$u(x,t) = x^2 + 2t$$

Example 4 [4]

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < \pi, \quad 0 < t \quad (13)$$

With boundary conditions

$$u(0,t) = 4 - e^{-t} \quad , \quad u(\pi,t) = 4 + e^{-t} \tag{14}$$

And initial conditions $u(x,0) = 4 - \cos\, x$

Solution:

The exact solution is $u(x,t) = 4 - e^{-t}\cos\, x$

Taking Elzaki Transform on both sides on (13)& (14)and using initial condition we get

$$v\frac{d^2T(x,v)}{dx^2} - T(x,v) = -v^2(4 - \cos x) \tag{15}$$

$$T(0,v) = 4v^2 - \frac{v^2}{1+v} \quad , \qquad T(\pi,v) = 4v^2 + \frac{v^2}{1+v} \tag{16}$$

After solving above differential equation (15) we get

$$T(x,v) = C_1 e^{\frac{x}{\sqrt{v}}} + C_2 e^{-\frac{x}{\sqrt{v}}} + 4v^2 - \frac{v^2 \cos x}{1+v}$$

Using (16) we get

$$T(x,v) = 4v^2 - \frac{v^2 \cos x}{1+v}$$

Taking inverse Elzaki Transform on both sides

$$u(x,t) = 4 - e^{-t}cosx$$

Conclusion

In this work, Elzaki Transform is used to solve 1-dimensional heat equation.This result has been extracted that Elzaki Transform plays a key role in finding the analytical solution for a wide class of initial boundary value problems.It is heavily used in solving PDEs and ODEs. The implementation of this method shows that it is easy to use, simple, time saving then the other existing methods.

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